1.1 Connected Spaces

Dfn 1.1 (i) a topological space X is *connected* if it cannot be written as a disjoint union of open subsets $X = A \sqcup B$.
(ii) X is *disconnected* if X is not connected.

Rem: X is connected $\Leftrightarrow$ the only clopen subsets of X are $\emptyset$ and X.

Thm 1.3: $\mathbb{R}$ with Euclidean topology is connected.

Thm 1.4: $f: X \to Y$ continuous, X connected, then $f(X)$ connected.

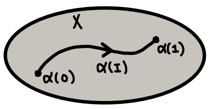
Cor 1.5: Connectedness is a topological invariant.

Prop 1.6: A connected, and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Cor 1.7: Every nonempty interval $J \subseteq \mathbb{R}$ is connected.

1.2 Paths and Path Components $\pi_0(X)$.

Dfn 1.8: A *path* in a topological space X is a cont. map $I = [0,1] \to X$. Its *initial point* is $\alpha(0) \in X$ and its *terminal point* is $\alpha(1) \in X$



Ex 1.9: $X \subseteq \mathbb{R}^n$ is convex if $\forall x_0, x_1 \in X$,
$[x_0, x_1] := \{(1-t)x_0 + tx_1 : 0 \le t \le 1\} \subset X$

Dfn 1.10: X is *path-connected* if $\forall\, x_0, x_1 \in X$, $\exists$ a path $\alpha: I \to X$ with $\alpha(0) = x_0$ to $\alpha(1) = x_1$.

Thm 1.12: path-connected $\Rightarrow$ connected.

Thm 1.14: $n \geq 2$, $\mathbb{R}^n \ncong \mathbb{R}$

Thm 1.16: any connected, open subset $\Omega \subseteq \mathbb{R}^n$ is connected.

Thm 1.17: $f: X \to Y$ cont., X path-connected. Then $f(X)$ is path-connected.

Cor 1.18: path-connectedness is a topological invariant.

Prop 1.19: For any equivalence relation on a path-connected space, $Y = X/\sim$ is also path connected.

Dfn 1.20: *path-relation:* $\forall x, y \in X$,
$x \sim y \Leftrightarrow \exists$ path $\alpha: I \to X$ s.t. $\alpha(0) = x$, $\alpha(1) = y$.

Dfn 1.21: *Constant path:* $\alpha: I \to X$; $\alpha: t \mapsto x$

Dfn 1.22: *reverse of path:* $-\alpha: I \to X$; $t \mapsto \alpha(1-t)$

Dfn 1.23: *Concatenation of paths:* $\alpha, \beta: I \to X$, with $\alpha(1) = \beta(0) = x$:

$$\alpha \bullet \beta: I \to X;\ t \mapsto \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Prop 1.24: path relation is an equivalence relation on X.

Dfn 1.25: (i) *path components* of X are the equivalence classes of the path relation:
$[x] = \{y \in Y : y \sim x\}$
(ii) *set of path components*: $\pi_0(X) := X/\sim$
The function $X \to \pi_0(X)$, $x \mapsto [x]$ is surjective.

WARNING: bijection $\pi_0(X) \to \pi_0(Y) \not\Rightarrow X \cong Y$.

Rem: a cont. $f: X \to Y$ induces a function $f_*: \pi_0(X) \to \pi_0(Y)$; $[x] \mapsto [f(x)]$.

Covariant: $(g \circ f)_* = g_* \circ f_*$.

Ex 1.28: X path connected $\Leftrightarrow$ $\pi_0(X)$ = trivial.

2. Homotopy Theory

Dfn 2.1: *Homotopy* between maps $f, g : X \to Y$ is a map $h : X \times I \to Y$ such that
$h(x,0) = f(x)$, $h(x,1) = g(x)$
we say then that f and g are *homotopic*, denoted
$h : f \simeq g : X \to Y$.

$Y^X :=$ Space of maps $X \to Y$

Rem: (i) $\forall\, t \in I$, $\exists$ map $h_t : X \to Y;\ x \mapsto h(x,t)$
(ii) $\forall\, x \in X$, $\exists$ path $f(x) \to g(x)$ $h(x,-)(t) = h(x,t)$.

Prop 2.3: homotopy defines an equivalence relation on Y^X.

Constant homotopy: $h : f \simeq f : X \to Y;\ (x,t) \mapsto f(x)$
reverse homotopy: $h : f \simeq g : X \to Y$, then
$-h : g \simeq f : X \to Y;\ (x,t) \mapsto h(x, 1-t)$

Concatenation: $h_1 : f_1 \simeq f_2 : X \to Y$, $h_2 : f_2 \simeq f_3 : X \to Y$,
then $h_1 \bullet h_2 : X \times I \to Y$;
$$(x,t) \mapsto \begin{cases} h_1(x, 2t) & 0 \le t \le 1/2 \\ h_2(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

Prop 2.4: $f, g : X \to Y$, $Y \subseteq \mathbb{R}^n$ convex, then $f \simeq g$ via
$h : X \times I \to Y;\ (x,t) \mapsto (1-t) f(x) + t\, g(x)$.

Dfn 2.6: X, Y are *homotopy equivalent* if $\exists$ maps
$f : X \to Y$, $g : Y \to X$ and homotopies
$h : gf \simeq 1_X : X \to X$, $k : fg \simeq 1_Y : Y \to Y$
we say f and g are *inverse homotopy equivalences*

Rem: homotopy equivalence of spaces is an equiv. relation.

Dfn 2.7: X *contractible* := homotopy equivalent to a point.

Rem: $f, g : X \to Y$, Y contractible $\Rightarrow f \simeq g$.

Prop 2.11: Homeomorphic spaces are homotopy equivalent.

Rem: Compactness is ignored by homotopy equivalence
e.g. non-compact $D^n / \{0\} \simeq S^{n-1}$

Prop 2.12: $f : X \to Y$ hom. equiv. induces bijection
$f_* : \pi_0(X) \to \pi_0(Y);\ [x] \mapsto [f(x)]$.

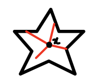
Rem: X path-connected, Y not, then $X \not\simeq Y$
Ex 2.13: $X = \{0\}$, $Y = \{0, 1\}$ with discrete topology.

Rem: two finite spaces with discrete topology are hom equiv iff they have the same # of elements.

Dfn 2.14: (i) *retraction* of X onto subspace $Y \subseteq X$ is a map $r : X \to Y$ s.t $r(y) = y\ \forall\, y \in Y$.
Y is called a *retract* of X.
(ii) *deformation retraction* of X onto Y is a map $h : X \times I \to X$ s.t.
$h(x,0) \in Y$, $h(x,1) = x\ \forall\, x \in X$,
$h(y,0) = y\ \ \forall\, y \in Y$.
Y is called a *deformation retract* of X.

Ex 2.17: $X \neq \emptyset$, *trivial topology* $\tau = \{\emptyset, X\}$.
Then any function $Y \to X$ is continuous. Then for $x_0 \in X$, $\{x_0\}$ is a deformation retract via
$$h : X \times I \to X\ ;\ (x,t) \mapsto \begin{cases} x_0 & t = 0 \\ x & 0 < t \le 1 \end{cases}$$

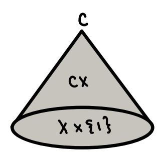
Dfn 2.18: $X \subseteq \mathbb{R}^n$ is *star-shaped* at $x \in X$ if $\forall\, y \in X$,
$[x,y] = \{(1-t)x + ty : 0 \le t \le 1\} \subseteq \mathbb{R}^n$
is contained in X



Rem: convex if star-shaped at all points.

Prop 2.20: $X \subseteq \mathbb{R}^n$ star-shaped at $x_0 \in X$, then $\{x_0\} \subseteq X$ is a deformation retract via
$h : X \times I \to X\ ;\ (x,t) \mapsto x_0 + t(x - x_0)$

Dfn 2.22: *cone* on $X \neq \emptyset$ is the quotient space obtained from $X \times I$ by collapsing $X \times \{0\}$ to a point:
$$CX = {X \times I}/{(X \times \{0\})}\ , \quad = {X \times I}/{\sim}\ , \quad (x,0) \sim (y,0).$$
$X = X \times \{1\} \subseteq CX$ is called the *cone base*.



path: $\alpha : I \to CX\ ;\ t \mapsto [x, st]$, $\alpha(0) = c$, $\alpha(1) = [x, t]$.

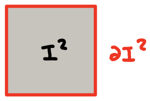
Prop 2.23: (i) $\{c\} \subset CX$ is a deformation retract of CX, so CX is contractible.
(ii) $f : X \to Y$ is hom to a constant map $g : X \to Y,\ x \mapsto y_0$
iff $\exists\ F : CX \to Y$ s.t $F[x,1] = f(x)$, $F[x,0] = y_0$.

Prop 2.25: extend $\sim$ on X to a relation R on CX by
$(x,s) R (y,t) \Leftrightarrow (x,s) = (y,t)$ or $s = t = 0$ or $s = t = 1$ and $x \sim y$
Then X/R is a deform retract of $Y = (CX \setminus \{c\})/R$,
and Y is hom equiv. to $X/\sim$.

3. Some Quotient Spaces of I^2 and ∂I^2

$I^2 = I \times I = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$

$\partial I^2 = \{(x,y) \in \mathbb{R}^2 \mid x=0 \text{ or } x=1 \text{ or } y=0 \text{ or } y=1\}$



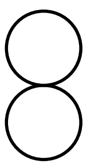
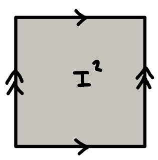
Let $X = \partial I^2$.

Rem: $\beta : CX \to I^2$;
$[(x,y),t] \mapsto (v,w) = (1-t)(\frac{1}{2},\frac{1}{2}) + t(x,y)$
is a homeomorphism $\beta(c) = (\frac{1}{2},\frac{1}{2})$,
$\beta([(x,y),1]) = (x,y)$

3.1 The Figure 8

one point union: for $x_0 \in X$, $y_0 \in Y$, define
$X \vee Y := (X \sqcup Y)/\sim, \quad x_0 \sim y_0$.

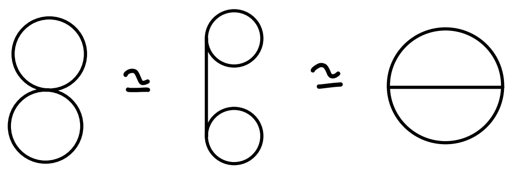
$8 = S^1 \vee S^1$

 $\cong$ 

$(x,0) \sim (x,1)$
$(0,y) \sim (1,y)$

$\partial I^2/\sim \;\to 8$; $\begin{cases} [x,0] = [x,1] \mapsto e^{2\pi i x} \text{ in } 1^{st}\ S^1 \\ [0,y] = [1,y] \mapsto e^{2\pi i y} \text{ in } 2^{nd}\ S^1 \end{cases}$

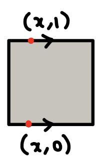
Rem:



$8 \qquad \simeq \qquad \Upsilon \qquad \simeq \qquad \theta$

3.2 The Cylinder

Cylinder $I \times S^1 = I^2/\sim$,
$(x,0) \sim (x,1)$

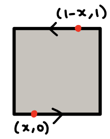


S^1 is deform. ret. of $I \times S^1$

punctured cylinder $\simeq 8$

3.3. Möbius Band

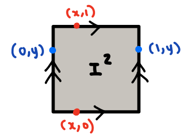
$M = I^2/\sim$, $(x,0) \sim (1-x,1)$



Contains circle as def. ret. $\quad S^1 \hookrightarrow M$; $[y] \mapsto [\frac{1}{2}, y]$.
Contains θ as def. ret., $\theta \hookrightarrow M$; $\theta = \partial I^2/\sim \;\to M \setminus \{(\frac{1}{2},\frac{1}{2})\}$.
punctured $M \simeq 8$.

3.4. The Torus

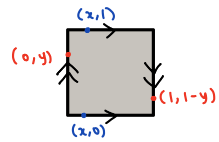
$S^1 \times S^1 = I^2/\sim$, $(x,0) \sim (x,1)$,
$(0,y) \sim (1,y)$



$\partial I^2/\sim \;\cong 8$

3.5 The Klein Bottle

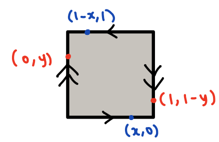
$K = I^2/\sim$, $(x,0) \sim (x,1)$,
$(0,y) \sim (1,1-y)$



8 deformation retract

3.6 Projective Plane

$\mathbb{RP}^2 = I^2/\sim$, $(x,0) \sim (1-x,1)$
$(0,y) \sim (1,1-y)$



$\partial I^2/\sim \;\cong S^1$

4. Cutting and Pasting Paths

Dfn 4.1. $f, g : A \to X$ maps, $B \subseteq A$ a subspace s.t
$$f(b) = g(b) \in X \quad (b \in B)$$
Then a **homotopy rel B** is a homotopy $h : f \simeq g : A \to X$ such that $h(b,t) = f(b) = g(b) \in X$

Idea: homotopy remains constant on B.

Ex 4.2: homotopy rel $\{0,1\}$ of two paths $\alpha_0, \alpha_1 : I \to X$ with the same endpoints: $\alpha_0(0) = \alpha_1(0)$, $\alpha_0(1) = \alpha_1(1)$, is a collection of paths $h_t : I \to X$ $(0 \leq t \leq 1)$ with same endpoints.
$$h_t(0) = \alpha_0(0) = \alpha_1(0), \quad h_t(1) = \alpha_0(1) = \alpha_1(1)$$

s.t. $h_0 = \alpha_0$, $h_1 = \alpha_1$, and $f : I \times I \to X, (s,t) \mapsto h_t(s)$ is cont.

Dfn 4.3: **concatenation of paths at $\lambda \in (0,1)$**. Take $\alpha, \beta : I \to X$ s.t $\alpha(1) = \beta(0) \in X$.
$$\alpha \bullet_\lambda \beta : I \to X; \; s \mapsto \begin{cases} \alpha\left(\frac{s}{\lambda}\right) & 0 \leq s \leq \lambda \\ \beta\left(\frac{s-\lambda}{1-\lambda}\right) & \lambda \leq s \leq 1 \end{cases}$$
$\alpha \bullet_\lambda \beta(0) = \alpha(0)$, $\alpha \bullet_\lambda \beta(\lambda) = \alpha(1) = \beta(0)$, $\alpha \bullet_\lambda \beta(1) = \beta(1)$.

Prop 4.5: rel $\{0,1\}$ homotopy class of $\alpha \bullet_\lambda \beta : I \to X$ is independent of λ

Dfn 4.6 + Prop 4.7: Can extend concatenation to finitely many paths, now λ is a partition of I, and concatenated path is independent of λ.

Ex 4.8: $\alpha, \beta, \gamma : I \to X$, $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$, $0 < \lambda < \mu < 1$:
$$\alpha \bullet_\lambda \beta \bullet_\mu \gamma : I \to X; \; s \mapsto \begin{cases} \alpha\left(\frac{s}{\lambda}\right) & 0 \leq s \leq \lambda \\ \beta\left(\frac{s-\lambda}{\mu-\lambda}\right) & \lambda \leq s \leq \mu \\ \gamma\left(\frac{s-\mu}{1-\mu}\right) & \mu \leq s \leq 1 \end{cases}$$

Rem: $\alpha \bullet -\alpha : s \mapsto \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ -\alpha(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$

$$: s \mapsto \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \alpha(2(s-1)) & 1/2 \leq s \leq 1 \end{cases}$$

So $\alpha \bullet -\alpha \simeq 1$ via homotopy: $\quad \alpha(0) = \alpha(1) = x_0$
$$h : (s,t) \mapsto \begin{cases} \alpha(2st) & 0 \leq s \leq 1/2 \\ \alpha(2t(s-1)) & 1/2 \leq s \leq 1 \end{cases}$$

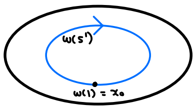
$h(s,0) = \alpha(0) = x_0$
$h(s,1) = \alpha \bullet -\alpha$

5. The Fundamental Group $\pi_1(X)$

Dfn 5.1 (i) A **based space** (X, x_0) is a space w/ a disting. base point $x_0 \in X$
(ii) A **based map** $f:(X,x_0) \to (Y,y_0)$ is a map $f: X \to Y$ such that $f(x_0) = y_0$
(iii) A **based homotopy** $h: f \simeq g : (X,x_0) \to (Y,y_0)$ is a homotopy $h: f \simeq g : X \to Y$ such that $h(x_0,t) = y_0$.

Prop 5.2. Based homot forms **equiv. rel** on all based maps

Dfn 5.3: **based loop** is based map $\omega: (S^1, 1) \to (X, x_0)$, where $1 := (1,0) \in S^1$



Equivalently...
Since $I/\{0 \sim 1\} = S^1$; $[t] \mapsto (\cos(2\pi t), \sin(2\pi t))$,
based loop = closed path $\alpha: I \to X$, $\alpha(0) = \alpha(1) = x_0$.
$\alpha(t) = \omega(\cos(2\pi t), \sin(2\pi t))$ $\qquad \alpha: I \xrightarrow{proj} S^1 \xrightarrow{\omega} X$

Rem: $\alpha, \beta : I \to X$ are homotopic iff $\alpha(I), \beta(I)$ are in the same path component of X.

Dfn 5.4: **Fundamental group** $\pi_1(X,x_0)$ at a base point $x_0 \in X$ is the set of rel $\{0,1\}$ homotopy classes $[\alpha]$ of closed paths $\alpha: I \to X$ s.t $\alpha(0) = \alpha(1) = x_0$.

- multiplication: $([\alpha],[\beta]) \mapsto [\alpha][\beta] := [\alpha \cdot \beta]$
- inverses: $[\alpha]^{-1} = [-\alpha]$
- identity: $[e_{x_0}]$ = constant path

Thm 5.5: $\pi_1(X,x_0)$ is a group

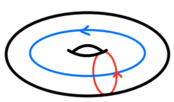
Rem 5.6:

X	$\mathbb{R}^n$	S^1	S^n $(n \geq 2)$	$\mathbb{RP}^n$ $(n \geq 2)$	$S^1 \times S^1$	8	M
$\pi_1(X)$	$\{1\}$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	$\mathbb{Z}$

Rem: loops in $\mathbb{R}^n$ are contractible : $h(s,t) = (1-t)\gamma(s)$

$\pi_1(\mathbb{RP}^n)$ is generated by square root loop:
$(\cos 2\pi t, \sin 2\pi t) \mapsto [\cos \pi t, \sin \pi t, 0, 0, \ldots]$

$\pi_1(S^1 \times S^1)$ gen by
$x \mapsto (x,1)$ and $y \mapsto (1,y)$



Dfn 5.7: (X,x_0) is **simply-connected** if X is path-connected and $\pi_1(X,x_0) = 1$

e.g. S^1 path-connected but not simply-connected.
Contractible space is simply-connected.

Prop 5.9 (i) map $f: X \to Y$ induces covariant group homo
$f_*: \pi_1(X,x_0) \to \pi_1(Y, f(x_0))$; $[\alpha] \mapsto [f(\alpha)]$.

$(gf)_* = g_* f_*$

Dfn 5.10 (X,x_0). $f: X \to Y$ is a **homotopy equivalence** rel $\{x_0\}$ if $\exists\, g: Y \to X$ s.t $g(f(x_0)) = x_0$, a homotopy rel $\{x_0\}$ $h: gf \simeq 1\!\!1_X : X \to X$, $h(x_0,t) = x_0$ and a homotopy rel $\{f(x_0)\}$ $k: fg \simeq 1\!\!1_Y : Y \to Y$, $k(f(x_0),t) = f(x_0)$.

Prop 5.11: $f_1 \simeq f_2 : X \to Y$, then $(f_1)_* = (f_2)_*$
(ii) $f: X \to Y$ a homo equiv rel $\{x_0\}$, then f_* an iso with inverse $(f_*)^{-1} = g_*$.

Ex 5.12: def retract $X \to Z$ gives a hom equiv rel Z.
inclusion $i: Z \to X$ hom. equiv. rel z_0 $\forall z_0 \in Z$
$\Rightarrow$ iso $i_*: \pi_1(Z,z_0) \to \pi_1(X,z_0)$.

Prop 5.13 (i) path $\gamma: I \to X$ determines an iso of groups.
$\gamma_\#: \pi_1(X,\gamma(0)) \to \pi_1(X,\gamma(1))$; $[\alpha] \mapsto [-\gamma \cdot \alpha \cdot \gamma]$

With inverse $(\gamma_\#)^{-1} = (-\gamma)_\#$
(ii) $\gamma_\#$ depends only on rel $\{0,1\}$ homotopy class of γ.
(iii) $\gamma = \gamma_1 \cdot \gamma_2$, then $\gamma_\# = (\gamma_2)_\# (\gamma_1)_\#$.

Ex 5.14: closed path $\gamma: I \to X$, $\gamma(0) = \gamma(1) = x_0 \in X$, determines conjugation automorphism:
$\gamma_\#: \pi_1(X,x_0) \to \pi_1(X,x_0)$; $[\alpha] \mapsto [-\gamma \cdot \alpha \cdot \gamma] = [\gamma]^{-1}[\alpha][\gamma]$.

Prop 5.15 $G, F: X \to Y$, $H: F \simeq G : X, Y$, $x_0 \in X$, define path
$\gamma: I \to Y$, $t \mapsto H(x_0, t)$

From $\gamma(0) = F(x_0)$ to $\gamma(1) = G(x_0)$, so we define an iso
$\gamma_\#: \pi_1(Y, F(x_0)) \to \pi_1(Y, G(x_0))$

Then the induced homomorphisms
$F_*: \pi_1(X,x_0) \to \pi_1(Y, F(x_0))$, $G_*: \pi_1(X,x_0) \to \pi_1(Y, G(x_0))$,
are such that $G_* = \gamma_\# F_*$

Thm 5.6: $f: X \to Y$ a hom. equiv, then
$f_*: \pi_1(X,x_0) \to \pi_1(Y, f(x_0))$
is an isomorphism of groups for any $x_0 \in X$.

6. Classification of Compact Surfaces

Dfn 6.1 (i) a Hausdorff topological space M is an **n-dimensional topological manifold** if it admits a countable open cover $\{U_\lambda\}_{\lambda\in\Lambda}$, each $U_\lambda \cong \mathbb{R}^n$

(ii) 2D manifold is called a **surface**.

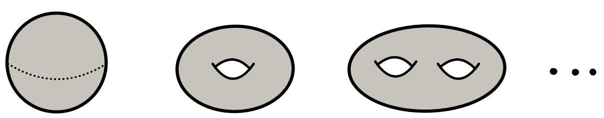
Dfn 6.3: a surface M is **orientable** if there is no subspace $N \subset M$ which is homeo to Möbius band.

Orientable: $\mathbb{R}^2$, S^2, $S^1 \times S^1$

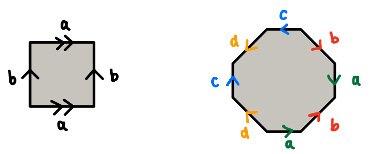
non orientable: Klein bottle, $\mathbb{R}P^n$)

Rem 6.5: $(M, \partial M)$ **surface with boundary**: $M \setminus \partial M$ is a surface (e.g. (D^2, S^1)).

Dfn 6.6: $g \geq 0 \in \mathbb{N}$. The **sphere with g handles**, $H(g)$, or the **g-holed torus**, is the orientable surface



H(0) H(1) H(2)



Dfn 6.8: $g \geq 1$, $g \in \mathbb{N}$. The **sphere with g cross-caps** $M(g)$ is the nonorientable surface obtained by taking S^2 and punching out g copies of D^2 and replacing each hole with a copy of the Möbius band

Thm 6.9 (Classification Thm for Compact Surfaces)

(1) Every connected, compact, orientable surface M is homeomorphic to $H(g)$ for some $g \geq 0$.

(2) Every connected, compact, nonorientable surface M is homeomorphic to $M(g)$ for some $g \geq 1$.

(3) **Connected, compact surfaces M, M' are homeomorphic iff $\exists$ a group isomorphism $\pi_1(M) \cong \pi_1(M')$.**

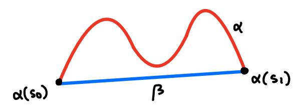
7. $\pi_1(S^n) = \{1\}$, $n \geq 2$

Lem 7.2: X compact metric space, $\{U_\lambda\}$ an open cover. Then $\exists$ $\delta > 0$, the **Lebesgue number of the cover**, s.t $\forall x \in X$, $B(x,\delta)$ lies entirely inside some single U_λ.

Prop 7.3 (i) Every map $\alpha: [s_0, s_1] \to \mathbb{R}^n$ is homotopic rel $\{s_0, s_1\}$ to a linear map

$$\beta : [s_0, s_1] \to \mathbb{R}^n ;\ s \mapsto \frac{(s_1 - s)\alpha(s_0) + (s - s_0)\alpha(s_1)}{s_1 - s_0}$$

With image $\beta([s_0,s_1]) = [\alpha(s_0), \alpha(s_1)] \subset \mathbb{R}^n$



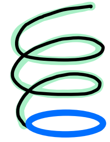
$H : I \times I ;$
$: (s,t) \mapsto (1-t)\alpha(s) + t\beta(s)$

(ii) $n \geq 2$, any path $\alpha : I \to S^n$ is homotopic rel $\{0,1\}$ to a path $\beta : I \to S^n$ which is not onto.

Thm 7.4: $\pi_1(S^n) = \{1\}$, $n \geq 2$.

Cor 7.5: $n \geq 3$, $\pi_1(\mathbb{R}^n \setminus \{0\}) = 1$.

8. $\pi_1(S^1) = \mathbb{Z}$



$S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

Prop 8.1. $\forall N \in \mathbb{Z}$, let $\omega_N : S^1 \to S^1 ;\ z \mapsto z^N$ be the map winding the circle around itself N times in the anticlockwise direction, with $\omega_N(1) = 1$. Then the function

$$\mathbb{Z} \to \pi_1(S^1) \ ;\ N \mapsto [\omega_N]$$

is a homomorphism of groups: $[\omega_N] = [\omega_1]^N$

Rem: $p : \mathbb{R} \to S^1 ;\ x \mapsto e^{2\pi i x}$ has the following properties:

① $\forall x \in \mathbb{R}$, $U = (x, x+1) \in \mathbb{R}$ is such that p restricts to a homeo

$$p| : U \to p(U) \ ;\ u \mapsto p(u)$$

with $p(U) = S^1 \setminus e^{2\pi i x}$

② p is onto, with $y = p(x)$, $\Rightarrow p^{-1}(y) = \{ x + n \mid n \in \mathbb{Z} \} \subset \mathbb{R}$

Dfn 8.2: *Lift of a map* $\alpha : [t_0, t_1] \to S^1$ is a map $\theta : [t_0, t_1] \to \mathbb{R}$ such that $\alpha(t) = e^{2\pi i \theta(t)} \in S^1$ $\qquad \alpha = p\theta : [t_0, t_1] \to S^1$.

With $\theta(t_0) = \theta_0 \in \mathbb{R}$ the *initial angle* and $\theta(t_1) = \theta_1 \in \mathbb{R}$ the *terminal angle*.

Angle map commutative diagram :

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[t_0, t_1]} \arrow[ur, "\theta"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$

Prop 8.3: $\alpha : [t_0, t_1] \to S^1$, $\theta_0 \in \mathbb{R}$ s.t $\alpha(t_0) = e^{2\pi i \theta_0}$, then $\exists!$ angle map $\theta : [t_0, t_1] \to \mathbb{R}$ for α with initial angle $\theta(t_0) = \theta_0 \in \mathbb{R}$.

Rem: angle maps are unique up to initial angle.
∴ possible initial angles differ by $\mathbb{Z}$.

Dfn 8.4. (i) The *degree* of a closed path $\alpha : I \to S^1$ is the difference between the terminal and initial angles in any angle map $\theta : I = [0,1] \to \mathbb{R}$ for α

$$\mathrm{degree}(\alpha) = \theta(1) - \theta(0) \in \mathbb{Z}$$

using $\alpha(0) = \alpha(1)$ to ensure this is an integer.

(ii) *degree of a loop* $\omega : S^1 \to S^1$ is

$$\mathrm{degree}(\omega) = \mathrm{degree}(\alpha) \in \mathbb{Z}$$

with α the closed path $\alpha : I \to S^1$, $t \mapsto \omega(e^{2\pi i t})$.

Ex 8.5: $\alpha_N : I \to S^1$, $t \mapsto e^{2\pi i N t}$; $\theta_N : I \to \mathbb{R}$, $t \mapsto Nt$ $\Big\}$ degree$(\alpha_N) = N$

antipodal map: $\omega : S^1 \to S^1 ;\ z \mapsto -z$; $\alpha : I \to S^1 ;\ t \mapsto e^{2\pi i (t + 1/2)}$; $\theta : I \to \mathbb{R} ;\ t \mapsto t + 1/2$ $\Big\}$ degree$(\alpha) = 1$

Prop 8.7: (i) $\gamma : \alpha \simeq \beta : [t_0, t_1] \to S^1$. Given θ_0 for α, $\exists!$ homotopy $\psi : \theta \simeq \phi : [t_0, t_1] \to \mathbb{R}$ such that

$$\gamma(s,t) = e^{2\pi i \psi(s,t)} \in S^1, \quad \psi(t_0, 0) = \theta_0$$

with $\theta : [t_0, t_1] \to \mathbb{R} ;\ s \mapsto \psi(s, 0)$,
$\phi : [t_0, t_1] \to \mathbb{R} ;\ s \mapsto \psi(s, 1)$

angle maps for α, β : $\alpha(s) = e^{2\pi i \theta(s)}$, $\beta(s) = e^{2\pi i \phi(s)}$.

(ii) If $\alpha(t_0) = \beta(t_0)$, $\alpha(t_1) = \beta(t_1)$ and $\gamma : \alpha \simeq \beta$ is a homotopy rel $\{t_0, t_1\}$, then ψ is a homotopy rel $\{t_0, t_1\}$.

Thm 8.8: $\pi_1(S^1) \cong \mathbb{Z}$, via group isomorphisms

$$\mathbb{Z} \to \pi_1(S^1) \ ;\ N \mapsto \omega_N$$

degree : $\pi_1(S^1) \to \mathbb{Z}$; $\omega \mapsto \mathrm{degree}(\omega)$

Note: $\alpha, \beta : I \to S^1$ closed paths, θ, ϕ their angle maps, $\theta(0) = \phi(0) = 0$. Then $\alpha \cdot \beta : I \to S^1$ has angle map

$$\lambda : I \to \mathbb{R} \ ;\ t \mapsto \begin{cases} \theta(2t) & 0 \le t \le 1/2 \\ \theta(1) + \phi(2t-1) & 1/2 \le t \le 1 \end{cases}$$

Also, $\deg(\alpha) = \theta(1)$, $\deg(\beta) = \phi(1)$, so that $\deg(\alpha \cdot \beta) = \deg(\alpha) + \deg(\beta)$.

Rem 8.11:
(i) two loops are homotopic iff $\deg(\omega) = \deg(\omega')$.
(ii) if a loop $\omega : S^1 \to S^1$ extends to a map $\delta\omega : D^2 \to S^1$, then $\deg(\omega) = 0$.

Dfn 8.13: The *winding number* around 0, $W(\alpha) \in \mathbb{Z}$, of a closed path $\alpha : I \to \mathbb{C} \setminus \{0\}$ $(\alpha(0) = \alpha(1))$

is defined to be the degree of the loop

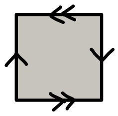
$$\omega : S^1 \to S^1 ;\ e^{2\pi i t} \mapsto \frac{\alpha(t)}{|\alpha(t)|}$$

that is, $W(\alpha) = \deg(\omega : S^1 \to S^1) = \theta(1) - \theta(0)$ for any angle map $\theta : I \to \mathbb{R}$ such that

$$\alpha(t) = |\alpha(t)| e^{2\pi i \theta(t)} \in \mathbb{C} \setminus \{0\}.$$

9. $\mathbb{RP}^n$

Dfn 9.1: $\mathbb{RP}^n = S^n/\sim$, $v \sim w \Leftrightarrow v = \pm w \in S^n$.
$p: v \mapsto [v]$, the natural projection, $p^{-1}([x]) = \{x, -x\}$.

Ex 9.3: $\mathbb{RP}^2 \cong$  $(x,0) \sim (1-x,1)$
$(0,y) \sim (1,1-y)$

Rem 9.2: $\mathbb{RP}^1 \cong S^1 \Rightarrow \pi_1(\mathbb{RP}^1) = \pi_1(S^1) = \mathbb{Z}$.

Rem: $\mathbb{RP}^n$ is compact and connected.

Dfn 9.5: *homogeneous coordinates*: $[x_0, \dots, x_{n+1}] \in \mathbb{RP}^n$,
projection: $S^n \to \mathbb{RP}^n$; $x \mapsto [x]$.

Dfn 9.6: The *square root loop* in $\mathbb{RP}^n$
$\sigma: S^1 \to \mathbb{RP}^n$; $z = \cos(2\pi t) + i\sin(2\pi t)$
$\mapsto \sqrt{z} = [\cos(2\pi t), \sin(2\pi t)]$

$\mathbb{Z}_2 = \{-1, +1\}$.

Prop 9.7: (i) $n \geq 1$, the square root loop $\sigma: S^1 \to \mathbb{RP}^1$ is a homeomorphism, so $\pi_1(\mathbb{RP}^1) = \pi_1(S^1) = \mathbb{Z}$.
(ii) The square of the square root $[\sigma] \in \pi_1(\mathbb{RP}^n)$ is
$[\sigma]^2 = [qp]$,
with $p: S^1 \to \mathbb{RP}^1$ the projection $v \mapsto [v]$, and
$q: \mathbb{RP}^1 \to \mathbb{RP}^n$; $[x_1, x_2] \mapsto [x_1, x_2, 0, 0, \dots, 0]$.
the inclusion
(iii) For $n \geq 2$, $[\sigma]^2 = 1$, and $\exists$ group homomorphism
$[\sigma]: \mathbb{Z}_2 \to \pi_1(\mathbb{RP}^n)$; $-1 \mapsto [\sigma]$.

Dfn 9.8: path-connected $U \subset \mathbb{RP}^n$ is *small* if $p^{-1}(U) \subset S^n$ has 2 path-components: U_+, U_-, and the restrictions
$p_+ = p| : U_+ \to U$, $p_- = p| : U_- \to U$
are homeomorphisms

Dfn 9.10: A *lift* of a map $\alpha: [t_0, t_1] \to \mathbb{RP}^n$ is a map $\theta: [t_0, t_1] \to S^n$ such that $\alpha(t) = p(\theta(t)) \in \mathbb{RP}^n$

$$\begin{tikzcd} & S^n \arrow[d, "p"] \\ {[t_0,t_1]} \arrow[ur, "\theta"] \arrow[r, "\alpha"] & \mathbb{RP}^n \end{tikzcd}$$

If α is closed, $p\theta(0) = \alpha(0) = \alpha(1) = p\theta(1) \in S$, so we define

$$\text{sign}(\alpha) = \begin{cases} +1 & \text{if } \theta(0) = \theta(1) \\ -1 & \text{if } \theta(0) = -\theta(1). \end{cases}$$

Prop 9.11: $\alpha: [t_0, t_1] \to \mathbb{RP}^n$, $\theta_0 \in S^n$ s.t. $\alpha(t_0) = p(\theta_0)$.
Then $\exists!$ lift $\theta: [t_0, t_1] \to S^n$ for α with initial point $\theta(t_0) = \theta_0$.

Thm 9.12 (i) The function $\text{sign}: \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2$;
$$: [\alpha] \mapsto \begin{cases} +1 & \theta(0) = \theta(1) \\ -1 & \theta(0) = \theta(-1) \end{cases}$$
is a surjective homomorphism with $\text{sign}([\sigma]) = -1$

(ii) $n = 1$, $\text{sign}: \pi_1(\mathbb{RP}^1) = \mathbb{Z} \to \mathbb{Z}_2$; $N \mapsto \begin{cases} +1 & N \text{ odd} \\ -1 & N \text{ even} \end{cases}$
is a surjection

(iii) $n \geq 2$, $\text{sign}: \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2$ is a group iso, with inverse $[\sigma]: \mathbb{Z}_2 \to \pi_1(\mathbb{RP}^n)$, $-1 \mapsto [\sigma]$.

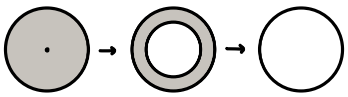
10. Fixed Points and Non-retraction

Dfn 10.1 a *fixed point* of a map $f: X \to X$ is a point $x \in X$ such that $f(x) = x$.

Brouwer Fixed point Theorem : every map $f: D^2 \to D^2$ has a fixed point, $n \geq 1$.

Dfn 10.3: $i: A \hookrightarrow X$ inclusion of $A \subset X$. The subspace A is a *retract* of X if $\exists$ a map $j: X \to A$ s.t $j \circ i: A \to A$ is the identity. The map j is a *retraction* of X onto A.

Ex 10.4: $S^{n-1} \subset D^n \setminus \{0\}$ is a retract.



$j: D^n \setminus \{0\} \to S^{n-1},$
$x \mapsto \frac{x}{\|x\|}.$

Prop 10.5: X path-connected, $i: A \to X$ inclusion of path connected subspace $A \subset X$.
(i) If A is a retract of X, $j: X \to A$ a retraction, then the induced group homos

$$i_*: \pi_1(A) \to \pi_1(X), \quad j_*: \pi_1(X) \to \pi_1(A)$$

are such that $j_* \circ i_*: \pi_1(A) \to \pi_1(A)$ is the identity
$\Rightarrow$ i_* injective, j_* surjective.
(ii) If $\pi_1(A) \neq \{1\}$ and $\pi_1(X) = \{1\}$ then A not retract of X.

Prop 10.7: If $\exists$ $f: D^n \to D^n$ without fixed points, then the inclusion $i: S^{n-1} \to D^n$ admits a retraction.

Thm 10.8 (non-retraction for $n = 1, 2$)
(i) $S^0 \subset D^1$ is not a retract
(ii) $S^1 \subset D^2$ is not a retract.

Cor 10.9 (BFPT, $n = 1, 2$)
Every map $f: D^2 \to D^2$ has a fixed point.

Prop 10.11: A map $f: S^n \to S^n$ without fixed points is homotopic to the antipodal map, $a: x \mapsto -x$.

homotopy : $S^n \times I \to S^n$; $(x, t) \mapsto \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$

Prop 10.3: (i) a map $f: S^1 \to S^1$ without fixed points has degree 1
(ii) a map $f: S^1 \to S^1$ with degree $\neq 1$ has a fixed point.

Rem: antipodal map has degree 1.

Recall... discrete: every subset is open.

11. Covering Spaces

General idea: compute $\pi_1(X,x_0)$ for a path-connected space X using the universal covering projection $p:\tilde{X}\to X$ with $\tilde{X}$ simply connected (path-connected and $\pi_1(\tilde{X})=0$), such that each $p^{-1}(x)\subset\tilde{X}$ is a copy of $\pi_1(X,x_0)$ as a discrete subspace.

Dfn 11.1: A *covering space* of a space X with *fibre* the discrete space F is a space $\tilde{X}$ with a *covering projection* map $p:\tilde{X}\to X$ such that for each $x\in X$, $\exists$ an open $U\subseteq X$, $x\in U$, and a homeomorphism

$$\phi: F\times U\to p^{-1}(U)$$

Such that $p\phi(a,u)=u\in U\subseteq X$. In particular, for each $x\in X$, $p^{-1}(x)$ is homeomorphic to F.

Dfn 11.2: Given a covering projection $p:\tilde{X}\to X$, let $\mathrm{Homeo}_p(\tilde{X})\leqslant\mathrm{Homeo}(\tilde{X})$ consisting of homeos $h:\tilde{X}\to\tilde{X}$ such that $ph=p:\tilde{X}\to X$ i.e. such that the diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Such h are called *covering translations*.

Dfn 11.4: A covering projection $p:\tilde{X}\to X$ with fibre F is *trivial* if $\exists$ a homeomorphism $\phi: F\times X\to\tilde{X}$ such that $p\phi(a,x)=x\in X$

Ex 11.5: F discrete, then $p:\tilde{X}:=F\times X\to X$; $(a,x)\mapsto x$ is trivial, with identity trivialization.

Ex 11.6 (ii) If p is trivial and $\phi_1,\phi_2: F\times X\to\tilde{X}$ are two trivializations, then $\exists!\ \sigma: X\to\mathrm{Homeo}(F)$ such that

$$\phi_2(a,x)=\phi_1(\sigma(x)(a),x)\in\tilde{X}$$

(iii) If p is trivial and ϕ is a trivialization, a covering translation $h:\tilde{X}\to\tilde{X}$ is then of the form

$$h:\tilde{X}\to\tilde{X};\ \phi(a,x)\mapsto\phi(\sigma(x)(a),x)$$

X path conn $\Rightarrow\sigma(x)$ constant, $\Rightarrow\mathrm{Homeo}_p(\tilde{X})=\mathrm{Homeo}(F)$

Rem: $h:X\to X$ is a homeomorphism iff h is a covering projection with fibre $F=\{1\}$.

Thm 11.8: Given a space $\tilde{X}$, and a $G\leqslant\mathrm{Homeo}(\tilde{X})$, define $\sim$ on $\tilde{X}$: $\tilde{x}_1\sim\tilde{x}_2\iff\exists\, g\in G$ s.t $\tilde{x}_2=g(\tilde{x}_1)$, $X:=\tilde{X}/\sim\ =\tilde{X}/G$, quotient $p:\tilde{X}\to X$.
Suppose that for each $\tilde{x}\in\tilde{X}$, $\exists$ open $U\subseteq\tilde{X}$, $\tilde{x}\in U$, Such that $g(U)\cap U=\emptyset\quad\forall g\in G\setminus\{1\}$.
Then $p:\tilde{X}\to X$ is a covering projection with fibre G.
$\tilde{X}$ path-conn $\Rightarrow X$ path-conn, and

$$\mathrm{Homeo}_p(\tilde{X})=G\subset\mathrm{Homeo}(\tilde{X}).$$

Dfn 11.10. $p:\tilde{X}\to X$ a covering projection. A *lift* of a map $f:Y\to X$ is a map $\tilde{f}:Y\to\tilde{X}$ such that

$$p(\tilde{f}(y))=f(y)$$

So there is a commutative diagram:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Ex 11.11: $p:\tilde{X}=F\times X\to X$, $f:Y\to X$. let $a\in F$, and define $\tilde{f}_a: Y\to\tilde{X}=F\times X$, $y\mapsto(a,f(y))$.

Y path connected $\Rightarrow$ every lift of f is of this type.

Thm 11.12: (Path lifting property) Let $p:\tilde{X}\to X$ be a covering projection, fibre F. Let $x_0\in X$, $\tilde{x}_0\in\tilde{X}$ s.t $p(\tilde{x}_0)=x_0$.
(i) $\forall\,\alpha: I\to X$, $\alpha(0)=x_0$, $\exists!$ lift $\tilde{\alpha}: I\to\tilde{X}$ s.t $\tilde{\alpha}(0)=\tilde{x}_0$.
(ii) $\alpha,\beta: I\to X$, $\alpha(0)=\beta(0)=x_0$, $\alpha(1)=\beta(1)\in X$. Then every rel $\{0,1\}$ homotopy $h:\alpha\simeq\beta: I\to\tilde{X}$ has a unique lift to a rel $\{0,1\}$ homotopy

$$\tilde{h}:\tilde{\alpha}\simeq\tilde{\beta}: I\to\tilde{X}$$

and in particular, $\tilde{\alpha}(1)=\tilde{h}(1,t)=\tilde{\beta}(1)\in\tilde{X}$.

Dfn 11.13: $p:\tilde{X}\to X$, $\alpha: I\to X$, the *fibre transport* bijection

$$\alpha_\#: p^{-1}(\alpha(0))\to p^{-1}(\alpha(1))\ ;\ \tilde{x}\mapsto\tilde{\alpha}_{\tilde{x}}(1),$$

where $\tilde{\alpha}_{\tilde{x}}: I\to X$ is the unique lift of α given by Thm 11.12 with $\tilde{\alpha}_{\tilde{x}}(0)=:\tilde{x}$

Prop 11.15: every covering projection $p:\tilde{X}\to X$ of a Simply-connected space is trivial.

Dfn 11.17: $p:\tilde{X}\to X$, X path-conn, is *universal* if $\tilde{X}$ is Simply-conn.

Thm 11.19: X path-conn, locally path-conn space with universal $p:\tilde{X}\to X$. Let $x_0\in X$, $\tilde{x}_0\in\tilde{X}$ s.t $p(\tilde{x}_0)=x_0$.
(i) function $\pi_1(X,x_0)\to p^{-1}(x_0)$; $[\alpha]\to\alpha_\#(x_0)$ is a bijection
(ii) $\forall\,[\alpha]\in\pi_1(X,x_0)$, $\exists!$ covering trans $h_\alpha\in\mathrm{Homeo}_p(\tilde{X})$ s.t $h_\alpha(\tilde{x}_0)=\alpha_\#(\tilde{x}_0)$
The function $\pi_1(X,x_0)\to\mathrm{Homeo}_p(\tilde{X})$; $[\alpha]\mapsto h_\alpha$ is an isomorphism of groups.